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# THE INTENSIFICATION OF MAGNETIC FIELDS IN PLASMAS

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## The Intensification of Magnetic Fields in Phasmas

R. K. Jaggi

### Figure Captions.

1. The deformed contour in the S-plane used to integrate the expression in equation (26).
2. Graph of  $H$  (Right hand side of equation 27) versus  $Z, \tau$  for some values of  $\tau$  and  $Z$  respectively.
3. Graph of  $V_+ (Z > 0), V_- (Z < 0)$  (Right hand side of equation 28) versus  $Z, \tau$  for some values of  $\tau$  and  $Z$  respectively.
4. Graph of  $E_+ (Z > 0), E_- (Z < 0)$  (Right hand side of equation 29) versus  $Z, \tau$  for some values of  $\tau$  and  $Z$  respectively.
5. Graph of  $H$  (Right hand side of equation 33) versus  $Z, \tau$  for some values of  $\tau$  and  $Z$  respectively.
6. Graph of  $E_-$  (Right hand side of equation 34 upper sign) versus  $Z, \tau$  for some values of  $\tau, Z$  respectively.
7. Graph of  $E_+$  (Right hand side of equation 35 lower sign) versus  $Z, \tau$  for some values of  $\tau, Z$  respectively.

## ABSTRACT

33632

Magnetohydrodynamic equations are used to obtain the solution to the problem of intensification of the magnetic field in an infinite plasma containing a uniform magnetic field by a motion of the semi-infinite plasma perpendicular to the field. Self consistent solution of the problem shows that a finite amount of magnetic field is produced in the direction of motion of the plasma. When the velocity of the medium is independent of time the magnetic field produced in the direction of motion is proportional to  $\sqrt{t}$  and any amount of magnetic field can be produced.

*author*

## The Intensification of Magnetic Fields in Plasmas

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The problem of the intensification of magnetic fields through the motion of a plasma already containing a magnetic field has important applications. In the sun the magnetic field of the sunspots and of bipolar magnetic regions may be produced in this way<sup>1,2</sup>. In the laboratory strong induction currents and associated magnetic fields are produced by shooting a beam of plasma transverse to a magnetic field<sup>3</sup>.

We shall consider two examples in which the plasma motion is initially perpendicular to the magnetic field and obtain expressions for the magnetic field produced in the direction of motion and the induced electric field produced perpendicular to both the given magnetic field and the velocity.

We shall use the magnetohydrodynamic equations for an incompressible medium of finite electrical conductivity:

$$\rho \frac{\partial \underline{v}}{\partial t} + \rho (\underline{v} \cdot \nabla) \underline{v} + \text{grad } p = \frac{1}{c} \underline{j} \times \underline{B} \quad (1)$$

$$\text{div } \underline{v} = 0 \quad (2)$$

$$\frac{\partial \underline{B}}{\partial t} = \text{Curl}(\underline{v} \times \underline{B}) + \frac{c^2}{4\pi\sigma} \nabla^2 \underline{B} \quad (3)$$

$$\text{div} \underline{B} = 0 \quad (4)$$

$$\text{Curl} \underline{E} = - \frac{1}{c} \frac{\partial \underline{B}}{\partial t} \quad (5)$$

where  $p$ ,  $\rho$ ,  $\underline{v}$ ,  $\underline{j}$ ,  $\underline{B}$  stand for the pressure, density, velocity vector, electric current and the magnetic field,  $\sigma$  is the electrical conductivity of the plasma.

In the first problem, we consider a uniform plasma occupying infinite space in the presence of a uniform magnetic in the  $z$ -direction (cartesian coordinates will be used.) A velocity  $\underline{v}_x = v_0 I(z)$  [where  $I(z) = 1$ , for  $z < 0$ ,  $I(z) = 0$  for  $z > 0$ ] is switched on suddenly at  $t = 0$ . We shall solve this initial value problem and obtain a solution for all  $z$  and for  $t > 0$ . Due to symmetry in the problem all quantities are functions of only the  $z$  coordinate (independent of  $x$ ,  $y$ ) and  $t$ . Equations (2), (4) immediately show that if  $\underline{v}_3$ ,  $B_3$  are the velocity and the magnetic field in the  $z$  direction

$$v_3 = \text{constant (Independent of } z) \quad (6)$$

$$B_3 = \text{constant (Independent of } z) \quad (7)$$

Equations (1), (3), (5) now yield in cartesian coordinates

$$\rho_0 \frac{\partial v_1}{\partial t} + \rho_0 v_3 \frac{\partial v_1}{\partial z} = \frac{B_3}{4\pi} \frac{\partial B_1}{\partial z} \quad (8)$$

$$\rho_0 \frac{\partial v_3}{\partial t} = -\frac{\partial}{\partial z} \left( p + \frac{B_1^2}{8\pi} \right) \quad (9)$$

$$\frac{\partial B_1}{\partial t} = v_3 \frac{\partial B_1}{\partial z} + \frac{c^2}{4\pi\sigma} \frac{\partial^2 B_1}{\partial z^2} \quad (10)$$

$$\frac{\partial B_3}{\partial t} = \frac{c^2}{4\pi\sigma} \frac{\partial B_3}{\partial z^2} \quad (11)$$

$$\frac{\partial E_2}{\partial z} = \frac{1}{c} \frac{\partial B_1}{\partial t} \quad (12)$$

The initial conditions of the problem are:

$$\begin{aligned} v_1 &= v_0 I(z) \\ B_1 &= 0 \\ B_3 &= B_0 \\ p &= p_0 \\ \rho &= \rho_0 \end{aligned} \quad (13)$$

At  $t = 0$ ;  $\left(\frac{\partial v_3}{\partial t}\right) = 0$  follows from (9). We can therefore assume

$v_3 = 0$ . From equation (11) we find that  $B_3 = B_0$  is the only solution satisfying equation (7).

Now using the dimensionless quantities

$$\begin{aligned} V &= \frac{v_1}{v_0}, \quad H = \frac{B_1}{B_0}, \quad Z = \frac{z}{\left(\frac{c^2}{4\pi\sigma v_0}\right)} \\ \tau &= \frac{t}{\left(\frac{c^2}{4\pi\sigma v_0^2}\right)}, \quad P = \frac{p}{(B_0^2/8\pi)}, \\ \lambda^2 &= \frac{B_0^2}{4\pi\rho_0 v_0^2}, \quad E = E_z / \left(\frac{v_0 B_0}{c}\right) \end{aligned} \tag{14}$$

Equation (8) - (10), (12) become

$$\frac{\partial V}{\partial \tau} = \lambda^2 \frac{\partial H}{\partial Z} \tag{15}$$

$$\frac{\partial}{\partial Z} [P + H^2] = 0 \tag{16}$$

$$\frac{\partial H}{\partial \tau} = \frac{\partial V}{\partial Z} + \frac{\partial^2 H}{\partial Z^2} \tag{17}$$

$$\frac{\partial E}{\partial Z} = \frac{\partial H}{\partial \tau} \tag{18}$$

We shall solve equation (15), (17), (18) subject to the initial

conditions  $H = 0$ ,  $V = I(Z)$ ;  $\partial V / \partial Z = -\delta(Z)$  (the

Dirac delta function). These give  $(\partial H / \partial \tau)_{\tau=0} = -\delta(Z)$   
from (17). We define the Laplace transform of  $V$ ,  $H$ ,  $E$  as

$$\begin{aligned}\bar{V}(Z, s) &= \int_0^\infty V(Z, \tau) e^{-s\tau} d\tau \\ \bar{H}(Z, s) &= \int_0^\infty H(Z, \tau) e^{-s\tau} d\tau \\ \bar{E}(Z, s) &= \int_0^\infty E(Z, \tau) e^{-s\tau} d\tau\end{aligned}\tag{19}$$

and obtain from (15), (17)

$$-I(Z) + s\bar{V} = \lambda^2 \frac{\partial \bar{H}}{\partial Z}\tag{20}$$

$$s\bar{H} = \frac{\partial \bar{V}}{\partial Z} + \frac{\partial^2 \bar{H}}{\partial Z^2}\tag{21}$$

Eliminating  $\bar{V}$  from (20), (21) we get

$$\frac{\partial^2 \bar{H}}{\partial Z^2} - \frac{s^2}{s + \lambda^2} \bar{H} - \frac{\delta(Z)}{s + \lambda^2} = 0\tag{22}$$



Now defining the Fourier transform of  $\bar{H}$  by

$$\begin{aligned} H(k, s) &= \int_{-\infty}^{\infty} dz e^{-ikz} \bar{H}(z, s) \\ \bar{H}(z, s) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikz} H(k, s) \end{aligned} \quad (23)$$

we obtain

$$H(k, s) = - \frac{1}{s + \lambda^2} \frac{1}{k^2 + \frac{s^2}{s + \lambda^2}} \quad (24)$$

From which, using (23) we obtain

$$\bar{H}(z, s) = - \frac{e^{-\frac{s|z|}{\sqrt{s + \lambda^2}}}}{2s \sqrt{s + \lambda^2}} \quad (25)$$

Taking the inverse Laplace transform we obtain

$$H(z, \tau) = - \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds \frac{e^{s\tau - \frac{s|z|}{\sqrt{s + \lambda^2}}}}{2s \sqrt{s + \lambda^2}} \quad (26)$$

where  $\epsilon = R_\epsilon(s)$  is a positive quantity. To evaluate this integral we deform the contour in the  $s$ -plane as shown in figure 1. The integral along the complete contour  $S$  is made up of contributions labelled as  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C'_2$  and  $C'_1$  in figure 1. Since  $s = 0$  is the only pole, the integral taken along the closed contour

is equal to  $2\pi i$  times the residue at  $s = 0$ , which equals

$-1/2\lambda$ . The integral along  $C_1, C_1'$  approaches zero as the radius of  $C_1, C_1' \rightarrow \infty$ . The integral along  $C_2 + C_2'$  yields

$$-\frac{1}{2\pi} \int_{\lambda^2}^{\infty} \frac{e^{-R\tau} \cos\left(\frac{Rz}{\sqrt{R-\lambda^2}}\right)}{R \sqrt{R-\lambda^2}} dR.$$

The contribution along  $C_3$  approaches  $-|z| e^{-\lambda^2\tau}$  as the radius of  $C_3$  approaches zero (See appendix I). We, therefore, obtain for  $H$  the expression

$$H(z, \tau) = -\frac{1}{2\lambda} + |z| e^{-\lambda^2\tau} + \frac{1}{2\pi} \int_{\lambda^2}^{\infty} \frac{e^{-R\tau} \cos\left(\frac{Rz}{\sqrt{R-\lambda^2}}\right)}{R \sqrt{R-\lambda^2}} dR. \quad (27)$$

Similarly we obtain

$$V(z, \tau) = \frac{1}{2} + e^{-\lambda^2\tau} + \frac{\lambda^2}{2\pi} \int_{\lambda^2}^{\infty} \frac{e^{-R\tau} \sin\left(\frac{Rz}{\sqrt{R-\lambda^2}}\right)}{R(R-\lambda^2)} dR \quad (28)$$

$$E(z, \tau) = \frac{1}{2} - \frac{\lambda^2 z^2}{2} e^{-\lambda^2\tau} - \frac{1}{2\pi} \int_{\lambda^2}^{\infty} \frac{e^{-R\tau} \sin\left(\frac{Rz}{\sqrt{R-\lambda^2}}\right)}{R} dR \quad (29)$$

where the upper sign applies for  $z > 0$  and the lower sign for  $z < 0$ .

After a sufficiently long time the last two terms in each of the equations (27), (28), (29) disappear and we obtain a uniform magnetic field  $-(1/2\lambda)$  along x-axis. The complete infinite medium moves with velocity of half the initial velocity of the semi-infinite medium and there is a uniform electric field  $(V_0 B_0/2\lambda)$  in the y direction.

Figure 2 shows a graph of H versus  $z$ ,  $\tau$  for some values of  $\tau$ ,  $z$  respectively. Similarly figures 3, 4 show some graphs of  $v(z)$ ,  $v(\tau)$  and  $E(z)$ ,  $E(\tau)$  respectively.

Let us now consider the case in which an external force acts on the semi-infinite medium  $z < 0$  and keeps it moving at a constant velocity  $v_0$  while the semi-infinite medium  $z > 0$  is held fixed. In this case the solution for the magnetic field is obtained by using equation (17) and then E is obtained from (18), According to our assumption  $V(z) = I(z)$ , and equation (17) yields

$$\frac{\partial H_1}{\partial \tau} - \frac{\partial^2 H_1}{\partial z^2} = \delta(z) \quad (30)$$

Using the Fourier Laplace transforms as before we obtain from equation (30)

$$H_1(k, s) = - \frac{1}{s(k^2 + s)} \quad (31)$$

and

$$H_1(z, s) = - \frac{e^{-|z|\sqrt{s}}}{2 s^{3/2}}$$

Taking the inverse Laplace transform we obtain

$$H_1(z, \tau) = - \sqrt{\frac{\tau}{\pi}} e^{-\frac{z^2}{4\tau}} + \frac{|z|}{2\sqrt{\pi}} \operatorname{erfc}\left(\frac{|z|}{2\sqrt{\tau}}\right) \quad (33)$$

Similarly we obtain

$$E(z, \tau) = \frac{1}{2} + \operatorname{erf}\left(\frac{|z|}{2\sqrt{\tau}}\right) \quad (34)$$

where

$$\operatorname{erf}(\alpha) = \frac{2}{\sqrt{\pi}} \int_0^{\alpha} e^{-x^2} dx, \quad \operatorname{erfc}(\alpha) = 1 - \operatorname{erf}(\alpha) \quad (35)$$

Using asymptotic expansion of the error function, in the limit  $|z|/\sqrt{\tau} \ll 1$  we obtain from (33)

$$H_1 = \sqrt{t/\pi} \quad (36)$$

or

$$B_1 = B_0 \sqrt{\frac{4 \sigma U_0^2 t}{c^2}} \quad (37)$$

For the case of sun  $\left(\frac{\sigma}{c^2}\right) \simeq 10^{-8} \text{ e.m.u.}$  Using

$U_0 \approx 10^4 \text{ cms/sec}$ ,  $t = 10^4 \text{ sec}$  we obtain  $B_1 \simeq 2000$  gauss. Thus it is possible to produce very strong magnetic fields as a result of plasma motions perpendicular to a given magnetic field.

Figure 5 shows a graph of  $H_1$  versus  $z$ ,  $\tau$  for some values of  $\tau, z$  respectively. Figures 6, 7 show graphs of  $E_-(z)$ ,  $E_-(\tau)$  and  $E_+(z)$ ,  $E_+(\tau)$  respectively.

## Discussion

The second example of this paper shows that in a plasma containing a magnetic field, a differential motion perpendicular to the field can produce large intensification of the magnetic field. The differential motion considered in this paper is of the form of a discontinuous jump. The magnitude of intensification will probably decrease if one considered a slow gradient in velocity.

In order to solve a problem which will be applicable to sunspots one has to consider a finite size blob of plasma moving perpendicular to a magnetic field. If one solves such a problem one finds that the blob comes to rest in a short time unless an external force continues to push it in its direction of motion.\* The assumption of constant velocity of the blob therefore needs justification. Unless one takes into account a force such as that of buoyancy, as done by Parker,<sup>1</sup> it is not clear how a pair of sunspots can be produced by the process considered by Romanchuk<sup>2</sup>.

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\* This result and some others will be published in a sequel to this paper.

# Appendix I

Consider the integral (26)

$$H(z, \tau) = - \frac{1}{4\pi i} \int \frac{e^{s\tau - \frac{s|z|}{\sqrt{s+\lambda^2}}}}{s\sqrt{s+\lambda^2}}$$

where  $s = -\lambda^2 + \lambda e^{i\theta}$ ,  $\pi/2 > \theta > -\pi$ . This integral

can be written as

$$\begin{aligned} H(z, \tau) &= \lim_{\lambda \rightarrow 0} \frac{e^{-\lambda^2 \tau}}{4\pi \lambda^2} \int_{\pi}^{\pi/2} e^{\frac{\lambda^2 |z|}{\sqrt{\lambda}} e^{-i\theta/2}} e^{i\theta/2} d\theta \\ &= \lim_{\alpha \rightarrow \infty} \frac{|z| e^{-\lambda^2 \tau}}{4\pi \alpha} \int_{\pi}^{-\pi} e^{i\theta/2} e^{\alpha e^{-i\theta/2}} d\theta \\ &= \lim_{\alpha \rightarrow \infty} \frac{|z| e^{-\lambda^2 \tau}}{4\pi \alpha} \left[ -4 \operatorname{Ci} \alpha - \alpha (2\pi + \operatorname{Si} \alpha) \right] \end{aligned}$$

where  $\operatorname{Si}(\alpha) = \int_0^{\alpha} \frac{\sin x}{x} dx$ .

Taking the limit  $\alpha \rightarrow \infty$  we obtain

$$H(z, \tau) = -|z| e^{-\lambda^2 \tau}$$

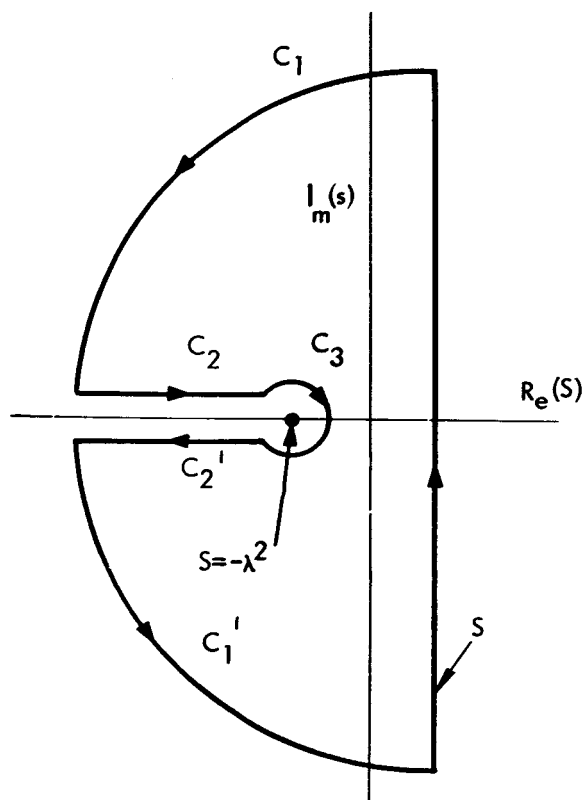
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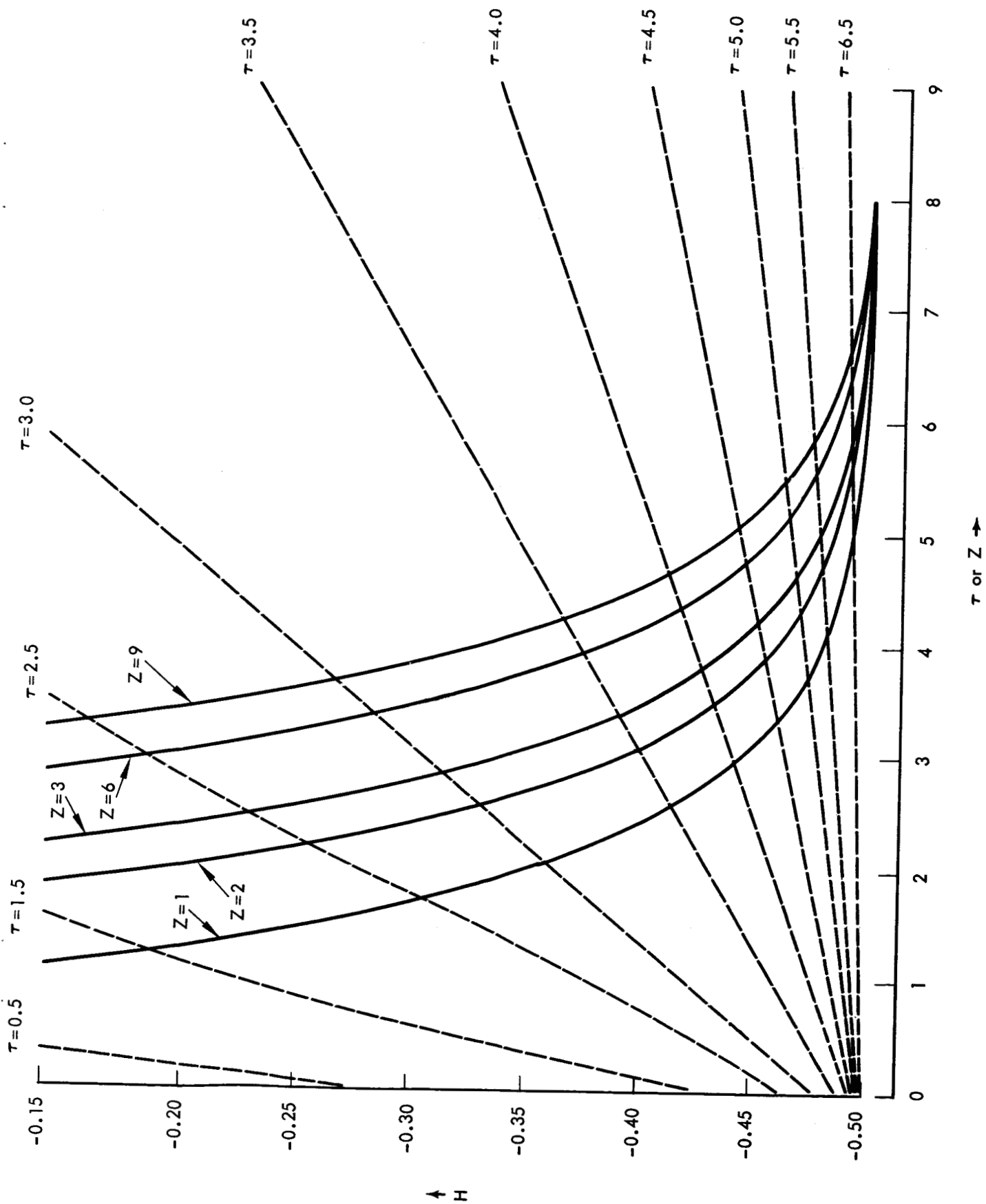
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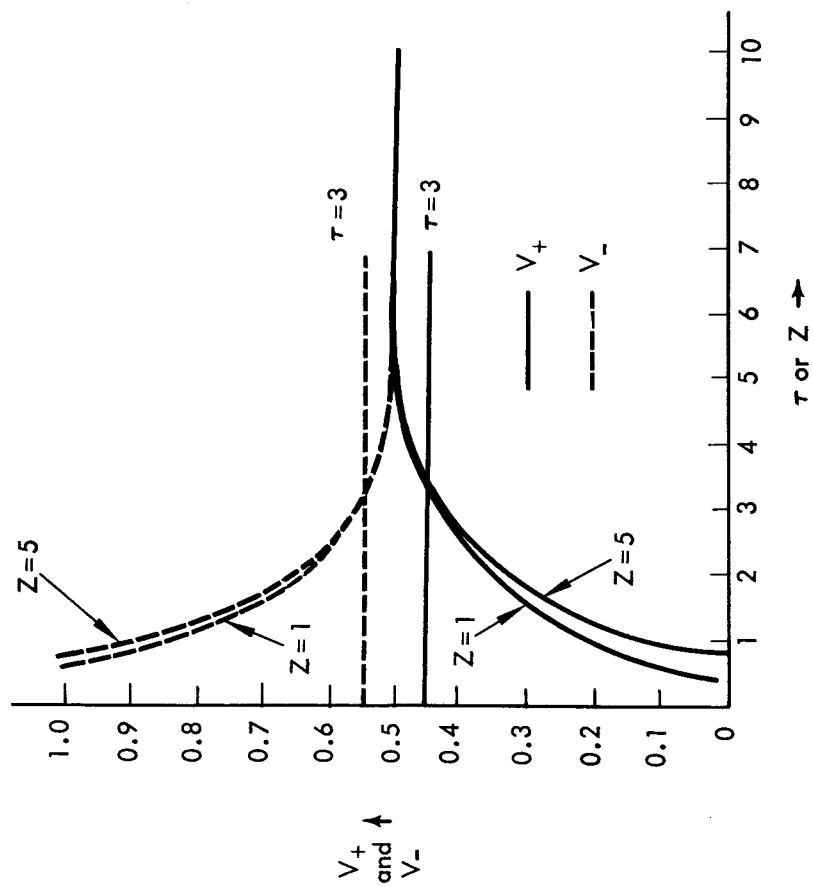


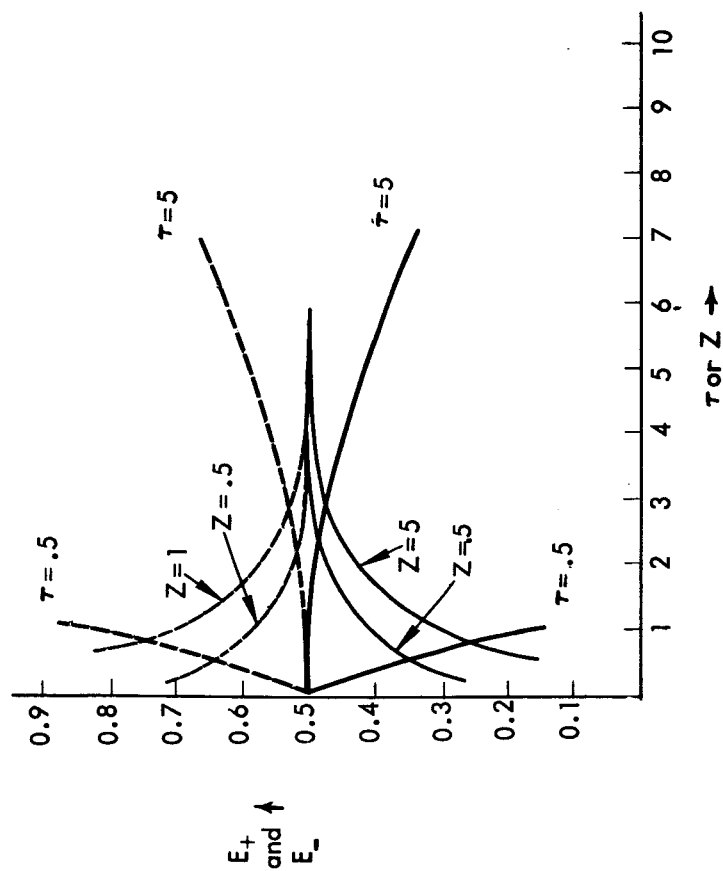
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—  $E_+(Z)$  for some values of  $\tau$  and  $E_+(\tau)$   
for some values of  $Z$ .

- - -  $E_-(Z)$  for some values of  $\tau$  and  $E_-(\tau)$   
for some values of  $Z$ .

